No percolation at criticality on certain groups of intermediate growth

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September 11, 2019

Abstract. We prove that critical percolation has no infinite clusters almost surely on any unimodular quasi-transitive graph satisfying a return probability upper bound of the form $p_n(v,v) \leq \exp\left[-\Omega(n^\gamma)\right]$ for some $\gamma > 1/2$. The result is new in the case that the graph is of intermediate volume growth.

1 Introduction

In **Bernoulli bond percolation**, first studied by Broadbent and Hammersley [11], each edge of a connected, locally finite graph G is either deleted or retained at random with retention probability $p \in [0,1]$, independently of all other edges. We denote the random graph obtained this way by ω_p. Connected components of ω_p are referred to as **clusters**. Percolation theorists are primarily interested in the geometry of the open clusters and how this geometry changes as the parameter p is varied. We are particularly interested in *phase transitions*, where the geometry of ω_p changes abruptly as we vary p through some special value. The first basic result about percolation, without which the model would not be nearly as interesting, is that for most infinite graphs (excluding e.g. one-dimensional counterexamples such as the infinite line graph $\mathbb{Z}$), percolation undergoes a *non-trivial phase transition*, meaning that the **critical probability**

$$p_c(G) = \inf\left\{p \in [0,1] : \omega_p \text{ has an infinite cluster almost surely}\right\}$$

is strictly between zero and one[1]. Indeed, a very general result to this effect has recently been proven by Duminil-Copin, Goswami, Raoufi, Severo, and Yadin [13], which implies in particular that $0 < p_c < 1$ for every quasi-transitive graph of superlinear volume growth.

Once we know that the phase transition is non-trivial, the next question is to determine what happens when p is exactly equal to the critical value p_c. This is a much more delicate question. Indeed, one of the most important open problems in percolation theory is to prove that critical percolation on the d-dimensional hypercubic lattice $\mathbb{Z}^d$ does not contain any infinite clusters almost

[1] It is a consequence of Kolmogorov's 0-1 law that the probability that ω_p contains an infinite cluster is either 0 or 1. Moreover, straightforward coupling arguments show that if ω_p contains an infinite cluster almost surely then ω_q contains an infinite cluster almost surley for every $p \leq q \leq 1$, see [22, Page 11].

surely for every $d \geq 2$. This problem was solved in two dimensions by Russo in 1981 [37], and for all $d \geq 19$ by Hara and Slade in 1994 [23]. More recently, Fitzner and van der Hoftstad [18] sharpened the methods of Hara and Slade to solve the problem for all $d \geq 11$. It is expected that this method can in principle, and with great effort and ingenuity, be pushed to handle all $d \geq 7$, while dimensions $3, 4, 5$, and 6 are expected to require new approaches. Similar results for other Euclidean lattices have been obtained in [5, 6, 14].

In their highly influential paper [10], Benjamini and Schramm proposed a systematic study of percolation on general **transitive** graphs, that is, graphs for which the action of the automorphism group on the vertex set has a single orbit (i.e., graphs for which any vertex can be mapped to any other vertex by a symmetry of the graph), and more generally on **quasi-transitive graphs**, for which there are only finitely many orbits. Prominent examples of transitive graphs include Cayley graphs of finitely generated groups. The following is among the most important of the many outstanding conjectures that they formulated.

Conjecture 1.1 (Benjamini and Schramm 1996). *Let G be a quasi-transitive graph. If $p_c(G) < 1$ then critical Bernoulli bond percolation on G has no infinite clusters almost surely.*

Aside from the previously mentioned results in the Euclidean setting, previous progress on Conjecture 1.1 can briefly be summarised as follows. Benjamini, Lyons, Peres, and Schramm [8] proved that Conjecture 1.1 holds for every *unimodular, nonamenable* transitive graph. Here, *unimodularity* is a technical condition that holds for every Cayley graph and every amenable quasi-transitive graph; see Section 2 for further background. Timár [40] later showed that critical percolation on any *nonunimodular* transitive graph cannot have *infinitely many* infinite clusters. Both results are easily generalised to the quasi-transitive setting. In [25], the second author of this article showed that critical percolation on any quasi-transitive graph of *exponential growth* cannot have a *unique* infinite cluster. Together with the aforementioned results of Benjamini, Lyons, Peres, and Schramm and Timár, this established that Conjecture 1.1 holds for every quasi-transitive graph of exponential growth. An alternative proof of this result in the unimodular case was recently given in [27]. All of these proofs have elements that are very specific to the exponential growth setting, and completely break down without this assumption.

In this paper, we build upon the ideas of [27] to develop a new method of proving that there are no infinite clusters at criticality. This new method applies in particular to certain transitive graphs of *intermediate growth*, for which the volume $|B(v, r)|$ of a ball of radius r grows faster than any polynomial in r but slower than any exponential of r. (In notation[2], a graph has intermediate growth if $r^{\omega(1)} \leq |B(v, r)| \leq e^{o(r)}$ as $r \to \infty$.) No such graph had previously been proven to satisfy Conjecture 1.1. The hypotheses of our results are most easily stated in terms of the n-step simple random walk return probabilities $p_n(v, v)$. Given $c > 0$ and $0 < \gamma \leq 1$, we say that a graph satisfies ($\mathrm{HK}_{\gamma,c}$) if

$$p_n(v, v) \leq \exp\left[-cn^{\gamma}\right] \qquad \text{for every } v \in V \text{ and } n \geq 1. \qquad (\mathrm{HK}_{\gamma,c})$$

We can now state our main theorem.

[2]Here we use Landau's asymptotic notation: In particular, for non-negative $f(n)$ and $g(n)$, "$f(n) = o(g(n))$ as $n \to \infty$" and "$g(n) = \omega(f(n))$ as $n \to \infty$" both mean that $\lim_{n\to\infty} f(n)/g(n) = 0$, while "$f(n) = O(g(n))$ as $n \to \infty$" and "$g(n) = \Omega(f(n))$ as $n \to \infty$" both mean that $\limsup_{n\to\infty} f(n)/g(n) < \infty$.

Theorem 1.2. *Let G be a unimodular quasi-transitive graph satisfying* ($\mathrm{HK}_{\gamma,c}$) *for some $c > 0$ and $\gamma > 1/2$. Then critical Bernoulli bond percolation on G has no infinite clusters almost surely.*

See Section 5 for a discussion of some variations on this result and a discussion of how our proof breaks down in the case $\gamma < 1/2$. Examples of groups of intermediate growth whose Cayley graphs satisfy the hypotheses of Theorem 1.2 can be constructed as *piecewise automatic groups* [16, Corollary 1] or using the notion of *diagonal products* [29]. (An analysis of the heat kernel on diagonal products will appear in a forthcoming work of Amir and Zheng.) Further examples can easily be constructed by, say, taking products of these groups with other groups of subexponential growth. For further background on groups of intermediate growth see [21] and references therein. Further works concerning probability on groups of intermediate growth include [17, 34, 36].

Note that Theorem 1.2 also implies that $p_c < 1$, so that we obtain an independent proof of the recent result of [13] in the special case of the class of graphs we consider. We also remark that Theorem 1.2 implies that there is no percolation at p_c on any unimodular quasi-transitive graph satisfying an isoperimetric inequality of the form $|\partial K| \geq c|K|/\log^{\delta}|K|$ for $c > 0$ and $0 < \delta < 1/2$, see [33] and Remark 3.3.

The proof of Theorem 1.2 is quantitative, and also yields explicit bounds on the tail of the volume of a critical cluster. In particular, we obtain the following bound in the transitive setting. The corresponding bound for quasi-transitive graphs is given in Theorem 4.1. We write $\mathbf{P}_p$ and $\mathbf{E}_p$ for probabilities and expectations taken with respect to the law of ω_p and write K_v for the cluster of v in ω_p.

Theorem 1.3. *Let $G = (V, E)$ be a unimodular transitive graph with maximum degree at most M satisfying* ($\mathrm{HK}_{\gamma,c}$) *for some $c > 0$ and $\gamma > 1/2$. Then for every $0 \leq \beta < (2\gamma - 1)/\gamma$ there exists $C(\beta) = C(\beta, \gamma, M, c)$ such that*

$$\mathbf{E}_p \exp\left[\log^{\beta}|K_v|\right] \leq C(\beta)$$

for every $p \leq p_c$.

We expect these bounds to be very far from optimal. Indeed, it is widely believed that critical percolation on any quasi-transitive graph of at least seven dimensional volume growth should satisfy $\mathbf{P}_{p_c}(|K_v| \geq n) \preceq n^{-1/2}$ as $n \to \infty$. See e.g. [24, 26, 28] and references therein for a detailed discussion of what is currently known regarding such bounds.

An immediate corollary of Theorem 1.3 is that Schramm's locality conjecture [9, Conjecture 1.2] holds in the case of graph sequences uniformly satisfying ($\mathrm{HK}_{\gamma,c}$) for some $\gamma > 1/2$.

Corollary 1.4. *Let $(G_n)_{n\geq 1}$ be a sequence of infinite unimodular transitive graphs converging locally to a transitive graph G, and suppose that there exist $c > 0$ and $\gamma > 1/2$ such that G_n satisfies* ($\mathrm{HK}_{\gamma,c}$) *for every $n \geq 1$. Then $p_c(G_n) \to p_c(G)$ as $n \to \infty$.*

See [9, 27] for a detailed discussion of this conjecture and for the definition of local convergence of graphs. The proof of Corollary 1.4 given Theorem 1.3 is very similar to the proof of [27, Corollary 5.1] and is omitted.

Proof overview The proof of Theorems 1.2 and 1.3 applies several of the ideas developed in the second author's recent paper [27], which we now review. Briefly, the methods of that paper allow us to convert bounds on the **two-point function** $\tau_p(u,v)$, defined to be the probability that u and v are connected in ω_p, into bounds on the tail of the volume of a cluster whenever $0 < p < p_c$. This is done as follows. For each set $K \subseteq V$, we write $E(K)$ for the set of edges of G that **touch** K, i.e., have at least one endpoint in K. For each edge e of G and $n \geq 1$, let $\mathscr{S}_{e,n}$ be the event that e is closed and that the endpoints of e are in distinct clusters each of which touches at least n edges and at least one of which is finite. The following universal inequality is proven in [27] using a variation on the methods of Aizenman, Kesten, and Newman [2]. It is a form of what we call the *two-ghost inequality*.

Theorem 1.5. *Let $G = (V, E)$ be a unimodular transitive graph of degree d. Then*

$$\mathbf{P}_p(\mathscr{S}_{e,n}) \leq 66d \left[\frac{1-p}{pn}\right]^{1/2}$$

for every $e \in E$, $p \in [0,1]$ and $n \geq 1$.

Next, an insertion-tolerance argument [27, Equation 4.2] is used to bound the tail of the volume in terms of the two-point function and the probability of $\mathscr{S}_{e,n}$ as follows. We define $\kappa_p(k) = \inf\{\tau_p(u,v) : u, v \in V, d(u,v) \leq k\}$, where $d(u,v)$ denotes the graph distance, and define $P_p(n) = \inf_{v\in V} \mathbf{P}_p(|E(K_v)| \geq n)$.

Lemma 1.6. *Let G be a connected, locally finite graph. Then*

$$P_p(n)^2 \leq \kappa_p(k) + \left[\sum_{i=0}^{k-1} p^{-i}\right] \sup_{e\in E} \mathbf{P}_p(\mathscr{S}_{e,n})$$

for every $0 \leq p < p_c$, $n \geq 1$ and $k \geq 1$.

Combining Theorem 1.5 and Lemma 1.6 allows us to convert bounds on $\kappa_p(k)$ into bounds on $P_p(n)$ when G is transitive and unimodular. For graphs of exponential growth, this was enough to conclude a bound of the form $P_{p_c}(n) \preceq n^{-\delta}$ using the exponential two-point function bound $\kappa_{p_c}(k) \leq \mathrm{gr}(G)^{-k}$ that was proven in [25].

In our setting, however, we do not have any non-trivial *a priori* control of the rate of decay of $\kappa_{p_c}(k)$. (Indeed, if we had such control we would already know that there is no percolation at p_c!) We circumvent this issue using the following bootstrapping procedure. We first prove via classical random walk techniques that if a transitive graph satisfies ($\mathrm{HK}_{\gamma,c}$) for some $c > 0$ and $0 < \gamma \leq 1$ then there exists $c' > 0$ such that the estimate

$$\mathrm{P}_{\mu_A}(X_k \in A) \leq \exp\left[-c' \min\left\{k^\gamma, \, \frac{k}{\log^\alpha |A|}\right\}\right]$$

holds for every finite set $A \subset V$ and $k \geq 0$, where $\alpha = (1-\gamma)/\gamma$ and P_{μ_A} denotes the law of the random walk $(X_k)_{k\geq 0}$ started from a uniformly random vertex of A. This is done in Section 3. Taking expectations, this gives in the transitive unimodular case that

$$\kappa_p(k) \leq \mathbf{E}_p\left[\mathrm{P}_\rho(X_k \in K_\rho)\right] = \mathbf{E}_p\left[\mathrm{P}_{\mu_{K_\rho}}(X_k \in K_\rho)\right] \leq \mathbf{E}_p \exp\left[-c' \min\left\{k^\gamma, \, \frac{k}{\log^\alpha |K_\rho|}\right\}\right],$$

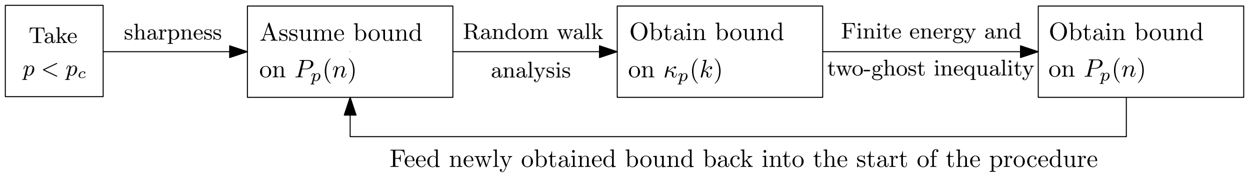

Figure 1: Schematic illustration of the bootstrapping procedure used (implicitly) in the proof of Theorem 1.3.

where the central equality follows from the mass-transport principle. Thus, we now have methods both for converting bounds on κ_p into bounds on P_p and vice versa, so that in particular we can convert one bound on $P_p(n)$ into another via an intermediate bound on $\kappa_p(k)$.

On the other hand, we know by sharpness of the phase transition [1, 15, 32] that $\mathbf{E}_p|K_\rho| < \infty$ for every $0 \leq p < p_c$, and consequently that for each $0 \leq p < p_c$ there exists a constant C_p such that $P_p(n) \leq C_p n^{-1}$ for every $n \geq 1$. To conclude the proof, it suffices to show that if we start with this bound and iteratively obtain new bounds on $P_p(n)$ using the above method, then in the case $\gamma > 1/2$ we obtain in the limit a bound on $P_p(n)$ that decays as $n \to \infty$ and holds uniformly on the whole range $0 \leq p < p_c$, as the same bound must then hold at p_c by an elementary continuity argument. See Figure 1 for a schematic outline. (A discussion of how this proof strategy breaks down in the case $\gamma < 1/2$ is given in Remark 5.5.) Rather than carrying out such a procedure explicitly, we instead use a similar method to prove a bound of the form

$$\mathbf{E}_p \exp\left[\log^\beta |K_\rho|\right] \leq C_\beta \sqrt{\mathbf{E}_p \exp\left[\log^\beta |K_\rho|\right]}$$

for each $p_c/2 \leq p < p_c$ and $0 \leq \beta < (2\gamma - 1)/\gamma$, which conveniently encapsulates this bootstrapping scheme and easily allows us to conclude the proof.

2 Background on unimodularity and the mass-transport principle

We now briefly review the notion of unimodularity and the mass-transport principle, referring the reader to [31, Chapter 8] for further background. Let $G = (V, E)$ be a connected, locally finite graph and let $\mathrm{Aut}(G)$ be the group of automorphisms of G. We write $[v] = \{\gamma v : v \in \mathrm{Aut}(G)\}$ for the orbit of a vertex $v \in V$ under $\mathrm{Aut}(G)$ and say that G is **unimodular** if $|\operatorname{Stab}_u v| = |\operatorname{Stab}_v u|$ for every $u, v \in V$ with $[u] = [v]$, where $\operatorname{Stab}_u = \{\gamma \in \mathrm{Aut}(G) : \gamma u = u\}$ is the stabilizer of u in $\mathrm{Aut}(G)$ and $\operatorname{Stab}_u v = \{\gamma v : \gamma \in \operatorname{Stab}_u\}$ is the orbit of v under Stab_u. Every Cayley graph and every amenable quasi-transitive graph is unimodular [39].

Suppose that G is a connected, locally finite, transitive unimodular graph. Then G satisfies the **mass-transport principle**, which states that for every $F : V^2 \to [0, \infty]$ that is diagonally-invariant in the sense that $F(\gamma u, \gamma v) = F(u, v)$ for every $u, v \in V$ and $\gamma \in \mathrm{Aut}(G)$, we have that

$$\sum_{v \in V} F(\rho, v) = \sum_{v \in V} F(v, \rho)$$

whenever ρ is an arbitrarily chosen root vertex of G. More generally, suppose that G is a connected, locally finite, *quasi-transitive* unimodular graph, and let $\mathcal{O} \subseteq V$ be a set of orbit representatives of the action of $\mathrm{Aut}(G)$. That is, $\mathcal{O}$ is such that for every $v \in V$ there exists a unique $o \in \mathcal{O}$ such that $[v] = [o]$. Then there exists a unique probability measure μ on $\mathcal{O}$ such that the identity

$$\sum_{o\in\mathcal{O}}\sum_{v\in V} F(o,v)\mu(o) = \sum_{o\in\mathcal{O}}\sum_{v\in V} F(v,o)\mu(o)$$

holds for every diagonally invariant $F : V^2 \to [0,\infty]$. In other words, if we choose a root $\rho \in V$ according to the measure μ then (G,ρ) is a *unimodular random rooted graph* in the sense of [4]. Similarly, if we choose ρ according to the degree-biased probability measure defined by

$$\tilde{\mu}(o) = \frac{\mu(o)\deg(o)}{\sum_{o'\in\mathcal{O}}\mu(o')\deg(o')} \qquad o \in \mathcal{O}$$

then the random (G,ρ) is a *reversible random rooted graph* in the sense of [7] (we will not make substantial use of these notions so we omit the definition). This gives rise to the following generalization of the two-ghost inequality to the quasi-transitive case, see [27, Remark 6.1].

Theorem 2.1. *Let $G = (V,E)$ be a connected, locally finite, unimodular quasi-transitive graph. Then*

$$\sum_{o\in\mathcal{O}}\mu(o)\sum_{e^-=o}\mathbf{P}_p(\mathscr{S}_{e,n}) \leq 66\left[\sum_{o\in\mathcal{O}}\mu(o)\deg(o)\right]\left[\frac{1-p}{pn}\right]^{1/2}$$

for every $p \in [0,1]$ and $n \geq 1$.

3 Random walk analysis

The goal of this section is to prove the following inequality regarding simple random walk on graphs satisfying ($\mathrm{HK}_{\gamma,c}$), which will play an important role in the proof of our main theorems. Given a locally finite graph $G = (V,E)$ and a finite set $D \subseteq V$, we write μ_D for the uniform measure on D. For each probability measure μ on V, we also write P_μ and E_μ for probabilities and expectations taken with respect to the law of a simple random walk $(X_k)_{k\geq 0}$ started at a vertex drawn from the measure μ.

Proposition 3.1. *Let $G = (V,E)$ be an infinite, connected graph with degrees bounded by M satisfying ($\mathrm{HK}_{\gamma,c}$) for some $c > 0$ and $0 < \gamma \leq 1$, and let $\alpha = (1-\gamma)/\gamma$. Then there exists a positive constant $c_1 = c_1(\gamma,c,M)$ such that*

$$\mathrm{P}_{\mu_D}\big(X_k \in D\big) \leq \left[\max_{u,v\in D}\frac{\deg(u)}{\deg(v)}\right]^{1/2}\exp\left[-c_1\min\left\{\frac{k}{\log^\alpha|D|}, k^\gamma\right\}\right] \tag{3.1}$$

for every finite set $D \subset V$ and every $k \geq 0$.

We expect that much of the content of this section will have been known as folklore by experts in random walks, but Proposition 3.1 has not, to our knowledge, previously appeared in the literature. Indeed, Proposition 3.1 will be deduced from a more general estimate, Corollary 3.8, which is a direct analogue in the infinite-volume setting of the L^∞ mixing time bounds of Goel, Montenegro, and Tetali [19].

The proof of Proposition 3.1 will apply the notion of the *spectral profile*, which we now introduce. Let $G = (V, E)$ be an infinite, connected, locally finite graph, and let P be the transition matrix of the simple random walk $(X_k)_{k=0}^{\infty}$ on G. For each finite set $A \subset V$ we define P_A to be the substochastic transition matrix of the random walk that is killed upon exiting A, which is given explicitly by $P_A(u, v) = P(u, v)\mathbb{1}(u, v \in A)$, and define $\lambda(A)$ to be the smallest eigenvalue of $I_A - P_A^2$, where $I_A(u, v) = \mathbb{1}(u = v, u \in A)$ and where we write P_A^i for $(P_A)^i$. Let π be the measure on V which assigns each v mass $\deg v$. We define the **spectral profile** of G to be the function $\Lambda : \mathbb{N} \to (0, 1]$ given by

$$\Lambda(L) := \inf\big\{\lambda(B) : B \subset V \text{ such that } \pi(B) \le L\big\}$$

if $L \ge \min_{v \in V} \pi(v)$ and $\Lambda(L) = 1$ otherwise. Given $c > 0$ and $\alpha \ge 0$, we say that a bounded degree graph satisfies ($\mathrm{SP}_{\alpha,c}$) if

$$\Lambda(x)^{-1} \le \frac{1}{c}\log^{\alpha}\left[\frac{x}{\max_v \pi(v)}\right] \qquad \text{for every } x \ge 2\max_v \pi(v). \tag{$\mathrm{SP}_{\alpha,c}$}$$

(The normalization by the maximal degree has been included in order to simplify various calculations below.)

Remark 3.2. We remark that our definition of the spectral profile is slightly non-standard. Indeed, when considering the *continuous-time* random walk, one considers the smallest eigenvalue of $I_A - P_A$ rather than of $I_A - P_A^2$ as we do here. It turns out however that using $I_A - P_A^2$ is more natural in the discrete-time setting. A simple application of the Perron-Frobenius theorem shows that the two definitions differ by at most a factor of two.

Remark 3.3. Many readers will be more familiar with the *isoperimetric profile* than with the spectral profile. We now briefly recall the relationship between these two profiles for their convenience; we will not apply the isoperimetric profile in the subsequent analysis. Let G be an infinite, locally finite graph. Its isoperimetric profile $(\Phi_*(x))_{x \ge 1}$ is defined to be

$$\Phi_*(x) = \inf\Big\{\frac{1}{\pi(A)}\sum_{a \in A, b \in V \setminus A} \pi(a)P(a, b) : A \subset V,\, \pi(A) \le x\Big\}.$$

A simple variation on Cheeger's inequality yields that

$$\frac{1}{4}\Phi_*^2(x) \le \Lambda(x) \le \Phi_*(x) \tag{3.2}$$

for every $x \ge \min_{v \in V} \pi(v)$; see the proof of [19, Lemma 2.44]. (Here we have a $1/4$ rather than the usual $1/2$ in the first inequality due to our nonstandard definition of Λ.)

The next proposition states that if $\alpha = (1 - \gamma)/\gamma$ then ($\mathrm{HK}_{\gamma,c}$) and ($\mathrm{SP}_{\alpha,c}$) are equivalent to within a controlled change of the constant c.

Proposition 3.4. *Let $0 < \gamma \le 1$ and let $\alpha = (1 - \gamma)/\gamma$. Then for every $c > 0$ and $M < \infty$ there exists $c_2 = c_2(\gamma, c, M)$ such that the following hold for every connected, locally finite graph G with maximum degree at most M:*

1. *If G satisfies ($\mathrm{HK}_{\gamma,c}$), then G satisfies (SP_{α,c_2}).*

2. If G satisfies ($\mathrm{SP}_{\alpha,c}$)*, then G satisfies* (HK_{γ,c_2})*.*

Proof. The first item is a special case of [38, Lemma 2.5], while the second item follows from [12, Proposition II.1]; see also [35, Section 2]. □

In light of this equivalence, it suffices to prove the following variation on Proposition 3.1.

Proposition 3.5. *Let G be an infinite, connected, bounded degree graph satisfying* ($\mathrm{SP}_{\alpha,c}$) *for some $\alpha \geq 0$ and $c > 0$. Then there exists a positive constant $c_1 = c_1(\alpha, c)$ such that*

$$\mathrm{P}_{\mu_D}[X_k \in D] \leq \left[\max_{u,v \in D} \frac{\pi(u)}{\pi(v)}\right]^{1/2} \exp\left[-c_1 \min\left\{\frac{k}{\log^\alpha |D|}, \, k^{1/(1+\alpha)}\right\}\right] \tag{3.3}$$

for every finite set $D \subset V$ and every $t \geq 0$.

Proposition 3.5 will in turn be deduced as a special case of the following proposition. We prove two variations on the same inequality: One of these bounds concerns random walk started at a uniform point of D, which is what arises in our analysis of percolation, while the other concerns random walk started at a point of D chosen according to the probability measure $\pi_D(v) = \pi(D)^{-1}\pi(v)\mathbb{1}(v \in D)$. This second bound is more natural from the random walk perspective, and we include it for future use since the proof is the same.

Proposition 3.6. *Let $G = (V, E)$ be a connected, locally finite graph with spectral profile Λ, and let $D \subseteq V$ be finite. If $\ell, k \geq 0$ satisfy*

$$k \geq \ell + 1 + \sum_{i=1}^{\ell} \frac{2\log 4}{\Lambda\big(4^{i+1} \max_{v\in D} \pi(v)|D|\big)} \qquad \text{then} \qquad \mathrm{P}_{\mu_D}(X_k \in D) \leq \left[\max_{u,v \in D} \frac{\pi(u)}{\pi(v)}\right]^{1/2} 2^{-\ell}. \tag{3.4}$$

Similarly, if $\ell, k \geq 0$ satisfy

$$k \geq \ell + 1 + \sum_{i=1}^{\ell} \frac{2\log 4}{\Lambda\big(4^{i+1}\pi(D)\big)} \qquad\qquad \text{then} \qquad \mathrm{P}_{\pi_D}(X_k \in D) \leq 2^{-\ell}. \tag{3.5}$$

We begin by introducing some basic notation. We identify each function $\phi \in \mathbb{R}^A$ with its extension to $\mathbb{R}^V$ obtained by setting $\phi \equiv 0$ on $V \setminus A$. For $i > 0$ and $\phi \in \mathbb{R}^V$ let $P_A^i \phi \in \mathbb{R}^A$ be given by

$$P_A^i \phi(u) := \sum_v P_A^i(u,v)\phi(v) = \mathrm{E}_u\Big[\phi(X_i)\mathbb{1}\big(T_{V\setminus A} > i\big)\Big],$$

where $T_{V\setminus A} = \inf\{k \geq 0 : X_k \in V \setminus A\}$ denotes the first time that the walk visits $V \setminus A$. Similarly, for each signed measure μ on V and $i > 0$ let μP_A^i be the signed measure supported on A given by

$$\mu P_A^i(u) := \sum_{v \in A} \mu(v) P_A^i(v,u) = \sum_{v\in A} \mu(v)\mathrm{P}_v\big(X_i = u, T_{V\setminus A} > i\big).$$

We also define $\langle \phi, \psi\rangle_\pi = \sum_{v\in V} \pi(v)\phi(v)\psi(v)$ for each $\phi, \psi \in \mathbb{R}^V$, and define $\|\phi\|_{2,\pi}^2 = \langle \phi,\phi\rangle_\pi$ and $\|\phi\|_{1,\pi} = \sum_{v\in V}\pi(v)|\phi(v)|$ for each $\phi \in \mathbb{R}^V$. Similarly, for each pair of signed measures μ, ν on

V we define $\langle \mu, \nu \rangle_{1/\pi} = \sum_{v \in V} \mu(v)\nu(v)/\pi(v)$ and define $\|\mu\|_{2,1/\pi}^2 = \langle \mu, \mu \rangle_{1/\pi}$. The Dirichlet form $\mathcal{E}_A : \mathbb{R}^V \to \mathbb{R}$ is defined by setting

$$\mathcal{E}_A(\phi) := \big\langle (I_A - P_A^2)\phi, \phi \big\rangle_\pi$$

for every $\phi \in \mathbb{R}^V$. It is a standard fact that $\lambda(A)$ can be expressed alternatively in terms of the Dirichlet form as

$$\lambda(A) = \inf \left\{ \frac{\mathcal{E}_A(\phi)}{\|\phi\|_{2,\pi}^2} : \phi \in \mathbb{R}_+^A,\ \phi \not\equiv 0 \right\}. \tag{3.6}$$

Indeed, this follows from [3, Theorem 3.33].

We note that the reversibility of P is inherited by P_A^k, so that for every $k \geq 0$ we have that $\pi(u)P_A^k(u,v) = \pi(v)P_A^k(v,u)$ for every $u, v \in V$ and $k \geq 0$. This is easily seen to imply that

$$\langle \mu P_A^t, \nu \rangle_{1/\pi} = \langle P_A^t \tfrac{\mu}{\pi}, \tfrac{\nu}{\pi} \rangle_\pi = \langle \tfrac{\mu}{\pi}, P_A^t \tfrac{\nu}{\pi} \rangle_\pi = \langle \mu, \nu P_A^t \rangle_{1/\pi} \tag{3.7}$$

for every pair of signed measures μ and ν, where $\frac{\mu}{\pi}$ and $\frac{\nu}{\pi}$ denote the functions $\frac{\mu}{\pi}(u) = \mu(u)/\pi(u)$ and $\frac{\nu}{\pi}(u) = \nu(u)/\pi(u)$ respectively.

The first step in the proof of Proposition 3.6 is the following key lemma, which is an analogue of [19, Lemma 2.1].

Lemma 3.7. *For every non-zero $\phi \in \mathbb{R}_+^A$ we have that*

$$\frac{\mathcal{E}_A(\phi)}{\|\phi\|_{2,\pi}^2} \geq \frac{1}{2}\Lambda\left(4\|\phi\|_{1,\pi}^2/\|\phi\|_{2,\pi}^2\right).$$

Proof. Let $\beta := \|\phi\|_{2,\pi}^2/4\|\phi\|_{1,\pi}$, and consider $B := \{v \in A : \phi(v) \geq \beta\}$. By Hölder's inequality and the fact that $\phi \geq 0$ we have that $\sup_v \phi(v) \geq \|\phi\|_{2,\pi}^2/\|\phi\|_{1,\pi}$, so that in particular the set B is not empty. On the other hand, we clearly have that $\pi(B) \leq \|\phi\|_{1,\pi}/\beta = 4\|\phi\|_{1,\pi}^2/\|\phi\|_{2,\pi}^2$. Defining the function $\psi := (\phi - \beta)\mathbb{1}_B$, we have that

$$\|\psi\|_{2,\pi}^2 \geq \left(\|\phi\|_{2,\pi}^2 - \|\phi^2\mathbb{1}_{A\setminus B}\|_{1,\pi}\right) - 2\beta\|\phi\mathbb{1}_B\|_{1,\pi} \geq \|\phi\|_{2,\pi}^2 - \beta\|u\mathbb{1}_{A\setminus B}\|_{1,\pi} - 2\beta\|\phi\mathbb{1}_B\|_{1,\pi} \geq \tfrac{1}{2}\|\phi\|_{2,\pi}^2,$$

where we used $\phi^2\mathbb{1}_{A\setminus B} \leq \beta\phi\mathbb{1}_{A\setminus B}$ and $2\beta\|\phi\|_1 = \frac{1}{2}\|\phi\|_{2,\pi}^2$. On the other hand, we have that

$$\begin{aligned}
\mathcal{E}_B(\psi) = \mathcal{E}_A(\psi) &= \frac{1}{2}\sum_{u,v \in B} \pi(u)P_A^2(u,v)(\phi(u) - \phi(v))^2 + \sum_{u \in B, v \in A\setminus B} \pi(u)P_A^2(u,v)(\phi(u) - \beta)^2 \\
&\leq \frac{1}{2}\sum_{u,v \in B} \pi(u)P_A^2(u,v)(\phi(u) - \phi(v))^2 + \sum_{u \in B, v \in A\setminus B} \pi(u)P_A^2(u,v)(\phi(u) - \phi(v))^2 \\
&\leq \mathcal{E}_A(\phi)
\end{aligned}$$

and hence that

$$\frac{\mathcal{E}_A(\phi)}{\|\phi\|_{2,\pi}^2} \geq \frac{1}{2}\frac{\mathcal{E}_B(\psi)}{\|\psi\|_{2,\pi}^2} \geq \frac{1}{2}\Lambda\Big(4\|\phi\|_{1,\pi}^2/\|\phi\|_{2,\pi}^2\Big),$$

where in the second inequality we used (3.6) and the fact that $\pi(B) \leq 4\|\phi\|_{1,\pi}^2/\|\phi\|_{2,\pi}^2$. $\square$

Corollary 3.8. *Let μ be a measure on V with $\mu(V) \le 1$. If $\ell, k \ge 0$ satisfy*

$$k \ge \ell + 1 + \sum_{i=1}^{\ell} \frac{2\log 4}{\Lambda(4^{i+1}\|\mu\|_{2,1/\pi}^{-2})} \qquad \textit{then} \qquad \|\mu P^k\|_{2,1/\pi} \le 2^{-\ell}\|\mu\|_{2,1/\pi}.$$

Proof. The claim holds vacuously if $\mu(V) = 0$, so suppose not. Let $A \subset V$ be finite with $\mu(A) > 0$, let $\mu_0 := \mu I_A$ be the restriction of μ to A, let $\mu_k := \mu_0 P_A^k$ for each $k \ge 1$, and let $\phi_k := P_A^k \frac{\mu}{\pi} \in \mathbb{R}_+^A$ for each $k \ge 0$. By (3.7) we have that

$$\|\mu_k\|_{2,1/\pi}^2 - \|\mu_{k+1}\|_{2,1/\pi}^2 = \|\phi_k\|_{2,\pi}^2 - \|\phi_{k+1}\|_{2,\pi}^2 = \langle \phi_k, \phi_k\rangle_\pi - \langle P_A^2\phi_k, \phi_k\rangle_\pi = \mathcal{E}_A(\phi_k). \tag{3.8}$$

Let $r_0 = 0$ and for each $\ell \ge 1$ let r_ℓ be maximal such that $\|\phi_{r_\ell}\|_{2,\pi}^2 > \|\phi\|_{2,\pi}^2/4^\ell$. Using the fact that the L_2 norm of ϕ_k is non-increasing in k, as well as (3.8) and Lemma 3.7, we deduce that

$$\begin{aligned}\|\phi_{k+1}\|_{2,\pi}^2 \le \|\phi_k\|_{2,\pi}^2 &\left[1 - \frac{1}{2}\Lambda\left(4\|\phi_k\|_{1,\pi}^2/\|\phi_k\|_{2,\pi}^2\right)\right] \\ &\le \|\phi_k\|_{2,\pi}^2\left[1 - \frac{1}{2}\Lambda\left(16/\|\phi_{r_{\ell-1}}\|_{2,\pi}^2\right)\right] \le \|\phi_k\|_{2,\pi}^2\left[1 - \frac{1}{2}\Lambda\left(4^{\ell+1}/\|\phi_0\|_{2,\pi}^2\right)\right]\end{aligned}$$

for every $\ell \ge 1$ and $r_{\ell-1} \le k \le r_\ell$, where we also used that $\|\phi_k\|_{1,\pi} = \mu_k(A) \le 1$ for every $k \ge 1$ in the second inequality. Thus, we have that

$$\|\phi_{r_{\ell-1}+1}\|_{2,\pi}^2 4^{-1} \le \|\phi\|_{2,\pi}^2 4^{-\ell} < \|\phi_{r_\ell}\|_{2,\pi}^2 \le \|\phi_{r_{\ell-1}+1}\|_{2,\pi}^2\left[1 - \frac{1}{2}\Lambda\left(4^{\ell+1}/\|\phi_0\|_{2,\pi}^2\right)\right]^{r_\ell - r_{\ell-1}-1}$$

for every $\ell \ge 1$. We deduce by an elementary calculation that $r_\ell - r_{\ell-1} - 1 < (2\log 4)/\Lambda(4^{\ell+1}/\|\phi_0\|_{2,\pi}^2)$. It follows immediately that if $k, \ell \ge 0$ satisfy

$$k \ge \ell + 1 + \sum_{i=1}^{\ell}\left\lfloor \frac{2\log 4}{\Lambda(4^{i+1}\|\mu I_A\|_{2,1/\pi}^{-2})}\right\rfloor \quad \text{then} \quad \|\mu P_A^k\|_{2,1/\pi} \le 2^{-\ell}\|\mu I_A\|_{2,1/\pi}.$$

The claim follows since the finite set A was arbitrary. □

We are now ready to prove Proposition 3.6.

Proof of Proposition 3.6. Let μ be a measure on V with $\mu(V) \le 1$, let $D \subseteq V$ be finite, and suppose that $k, \ell \ge 0$ are such that

$$k \ge \ell + 1 + \sum_{i=1}^{\ell} \frac{2\log 4}{\Lambda(4^{i+1}\|\mu\|_{2,1/\pi}^{-2})}. \tag{3.9}$$

Observe that for any measure μ on V we have by Cauchy-Schwarz that $\mu(D)^2 = \langle \pi \mathbb{1}_D, \mu\rangle_{1/\pi}^2 \le \pi(D)\|\mu\|_{2,1/\pi}^2$, and applying Corollary 3.8 we deduce that

$$\mu P^k(D)^2 \le \pi(D)\|\mu\|_{2,1/\pi}^2 4^{-\ell}$$

for every measure μ on V with $\sum_{v\in V}\mu(v) \leq 1$. We conclude by applying this estimate to the uniform distribution μ_D on D and the normalized stationary measure π_D on D and noting that $\|\mu_D\|_{2,1/\pi}^{-2} \leq [\max_{v\in D}\pi(v)]|D|$, that $\pi(D)\|\mu_D\|_{2,1/\pi}^2 \leq \max_{u,v\in D}\pi(u)/\pi(v)$, and that $\|\pi_D\|_{2,1/\pi}^2 = \pi(D)^{-1}$. □

We now perform the calculation required to deduce Proposition 3.5 from Proposition 3.6.

Proof of Proposition 3.5. By Proposition 3.6 and the assumption that G satisfies ($\mathrm{SP}_{\alpha,c}$), we have that if $k,\ell \geq 0$ satisfy

$$k \geq F(\ell) := \ell + 1 + \sum_{i=1}^{\ell} \frac{2\log 4}{c}\log^{\alpha}\left[4^{i+1}|D|\right] \qquad \text{then} \qquad \mathrm{P}_{\mu_D}(X_k \in D) \leq \left[\max_{u,v\in D}\frac{\pi(u)}{\pi(v)}\right]^{1/2} 2^{-\ell}.$$

Moreover, we clearly have that

$$F(\ell) \leq (\ell+1)\left[1 + \frac{2\log 4}{c}\log^{\alpha}(4^{\ell+1}|D|)\right] \leq C\max\big\{(\ell+1)^{1+\alpha},\, (\ell+1)\log^{\alpha}|D|\big\}$$

for some constant $C = C(c,\alpha)$, and hence that if $k,\ell \geq 0$ satisfy

$$k \geq C\max\big\{(\ell+1)^{1+\alpha},\, (\ell+1)\log^{\alpha}|D|\big\} \qquad \text{then} \qquad \mathrm{P}_{\mu_D}(X_k \in D) \leq \left[\max_{u,v\in D}\frac{\pi(u)}{\pi(v)}\right]^{1/2} 2^{-\ell}.$$

The result now follows by an elementary calculation. □

Proof of Proposition 3.1. This follows immediately from Propositions 3.4 and 3.5. □

4 Proofs of the main theorems

In this section we deduce Theorems 1.2 and 1.3 from Theorem 2.1, Lemma 1.6, and Proposition 3.1. We first formulate a generalization of Theorem 1.3 to the quasi-transitive case, which will then imply both Theorems 1.2 and 1.3. The statement of this generalization will employ the following quantitative notion of quasi-transitivity. Let $G = (V,E)$ be a unimodular quasi-transitive graph, let $\mathcal{O}$ be a complete set of orbit representatives for the action of $\mathrm{Aut}(G)$ on V, and let μ be as in Section 2. Given $r < \infty$ and $\varepsilon > 0$, we say that G satisfies ($\mathrm{QT}_{r,\varepsilon}$) if

$$\mu(o) \geq \varepsilon \text{ for every } o \in \mathcal{O}, \text{ and for every } u,v \in V \\ \text{there exist } w \in [u],\ z \in [v] \text{ such that } d(w,z) \leq r. \tag{$\mathrm{QT}_{r,\varepsilon}$}$$

Every unimodular transitive graph trivially satisfies ($\mathrm{QT}_{r,\varepsilon}$) with $r = 0$ and $\varepsilon = 1$, while every unimodular quasi-transitive graph satisfies ($\mathrm{QT}_{r,\varepsilon}$) for *some* $r < \infty$ and $\varepsilon > 0$, so that Theorems 1.2 and 1.3 both follow immediately from the following theorem.

Theorem 4.1. *Let $G = (V,E)$ be a unimodular quasi-transitive graph of maximum degree at most M satisfying ($\mathrm{HK}_{\gamma,c}$) and ($\mathrm{QT}_{r,\varepsilon}$) for some $r < \infty$, $c,\varepsilon > 0$ and $\gamma > 1/2$. Then for every*

$0 \le \beta < (2\gamma-1)/\gamma$ *there exists a constant* $K(\beta) = K(\beta,\gamma,c,r,\varepsilon,M)$ *such that*

$$\mathbf{E}_p \exp\left[\log^\beta |K_v|\right] \le K(\beta)$$

for every $v \in V$ *and* $p \le p_c$.

Given a unimodular quasi-transitive graph $G = (V,E)$, we let ρ be a random root vertex of G chosen according to the measure μ, and write $\mathbb{P}_p$ and $\mathbb{E}_p$ for probabilities and expectations taken with respect to the joint law of ω_p and ρ. Recall that we also write P_v for the law of a simple random walk on G started at the vertex v, and for each finite set $D \subset V$ we write P_{μ_D} for the law of a simple random walk started from a uniform point of D. (The two uses of μ should not cause confusion.)

Lemma 4.2. *Let* $G = (V,E)$ *be a unimodular quasi-transitive graph with degrees bounded by* M *satisfying* ($\mathrm{HK}_{\gamma,c}$) *for some* $c > 0$ *and* $0 < \gamma \le 1$, *and let* $\alpha = (1-\gamma)/\gamma$. *Then for every* $\beta \in (0,1]$ *there exists a constant* $c_2(\beta) = c_2(\beta,\alpha,c,M)$, *such that*

$$\mathbb{E}_p\left[\mathrm{P}_\rho(X_k \in K_\rho)\right] \le 2\left[\max_{u,v\in V}\frac{\deg(u)}{\deg(v)}\right]^{1/2}\mathbb{E}_p \exp\left[\log^\beta|K_\rho|\right]\exp\left[-c_2(\beta)k^{\beta/(\alpha+\beta)}\right].$$

for every $k \ge 1$ *and* $0 \le p \le 1$.

Proof. The claim is trivial if p is such that $|K_\rho| = \infty$ with positive probability, so suppose not. Let $c_1 = c_1(\gamma,c,M)$ be the constant from Proposition 3.1. Applying the mass-transport principle to the function $f : V^2 \to [0,\infty]$ defined by

$$f(u,v) = \mathbf{E}_p\left[\frac{\mathbb{1}(v \in K_u)}{|K_u|}\mathrm{P}_u[X_k \in K_u]\right],$$

we deduce that

$$\mathbb{E}_p\left[\mathrm{P}_\rho(X_k \in K_\rho)\right] = \mathbb{E}\sum_{v\in V} f(\rho,v) = \mathbb{E}\sum_{v\in V} f(v,\rho) = \mathbb{E}_p\left[\mathrm{P}_{\mu_{K_\rho}}(X_k \in K_\rho)\right],$$

and we deduce from Proposition 3.1 that

$$\begin{aligned}\mathbb{E}_p\left[\mathrm{P}_\rho(X_k \in K_\rho)\right] &\le \left[\max_{u,v\in V}\frac{\deg(u)}{\deg(v)}\right]^{1/2}\mathbb{E}_p\exp\left[-c_1\min\left\{\frac{k}{\log^\alpha|K_\rho|}, k^\gamma\right\}\right]\\ &\le \left[\max_{u,v\in V}\frac{\deg(u)}{\deg(v)}\right]^{1/2}\left[\mathbb{E}_p\exp\left[-c_1\frac{k}{\log^\alpha|K_\rho|}\right] + e^{-c_1k^\gamma}\right]. \quad (4.1)\end{aligned}$$

Using the inequality $\mathbb{E}_p[g(|K_\rho|)] \le \sup_{x\ge1}[g(x)/h(x)]\mathbb{E}_p[h(|K_\rho|)]$, which holds for every $g,h : \mathbb{N} \to \mathbb{R}_+$, we deduce that

$$\mathbb{E}_p\exp\left[-c_1\frac{k}{\log^\alpha|K_\rho|}\right] \le \sup_{x\ge1}\exp\left[-\log^\beta x - c_1\frac{k}{\log^\alpha x}\right]\mathbb{E}_p\exp\left[\log^\beta|K_\rho|\right].$$

A direct and elementary calculation shows that the minimum of $\log^\beta x + c_1 k \log^{-\alpha} x$ is attained when $\log^{\alpha+\beta} x = \alpha c_1 k/\beta$, and we deduce that there exists a constant $c = c(\alpha,\beta,c_1)$ such that

$$\mathbb{E}_p \exp\left[-c_1 \frac{k}{\log^\alpha |K_\rho|}\right] \le \mathbb{E}_p \exp\left[\log^\beta |K_\rho|\right] \exp\left[-c k^{\beta/(\alpha+\beta)}\right]. \tag{4.2}$$

Taking $c_2 = \min\{c_1, c\}$, the proof is now easily concluded by combining (4.1) and (4.2) and noting that $\gamma = 1/(\alpha+1) \ge \beta/(\alpha+\beta)$ and that $\mathbb{E}_p \exp\left[\log^\beta |K_\rho|\right] \ge 1$. □

Proof of Theorem 4.1. Let $\alpha = (1-\gamma)/\gamma$ and let $0 < \beta < (2\gamma-1)/\gamma = 1-\alpha$. Note that such a β exists precisely when $\gamma > 1/2$. Recall that M is a constant satisfying $\max_{v\in V} \deg(v) \le M$. Theorem 2.1 immediately implies that there exists a constant $C = C(M, r, \varepsilon)$ such that

$$\sup_{e\in E} \mathbf{P}_p(\mathscr{S}_{e,n}) \le C \left[\frac{1-p}{pn}\right]^{1/2}. \tag{4.3}$$

for every $p \in [0,1]$ and $n \ge 1$, where the event $\mathscr{S}_{e,n}$ is defined as in the introduction. Recall that we define $P_p(n) = \inf_{v\in V} \mathbf{P}_p(|E(K_v)| \ge n)$ and $\kappa_p(k) = \inf\{\tau_p(u,v) : d(u,v) \le k\}$ for each $p \in [0,1]$ and $n, k \ge 1$. Note also that we have the elementary bound $p_c \ge 1/(\max_{v\in V} \deg(v) - 1) > 1/2M$.

We have trivially that $\kappa_p(k) \le \mathbb{E}_p[\mathrm{P}_\rho(X_k \in K_\rho)]$ for every $0 \le p \le 1$ and $k \ge 1$. Thus, applying Lemmas 4.2 and 1.6 and rearranging, we obtain that

$$\begin{aligned} P_p(n)^2 &\le C \left[\sum_{i=0}^{k-1} p^{-i}\right] \left[\frac{1-p}{pn}\right]^{1/2} + \kappa_p(k) \\ &\le C \left[\sum_{i=0}^{k-1} p^{-i}\right] \left[\frac{1-p}{pn}\right]^{1/2} + 2\left[\max_{u,v\in V} \frac{\deg(u)}{\deg(v)}\right]^{1/2} \mathbb{E}_p \exp\left[\log^\beta |K_\rho|\right] \exp\left[-c_2(\beta) k^{\beta/(\alpha+\beta)}\right] \end{aligned}$$

for every $0 \le p < p_c$ and $k, n \ge 1$. Taking $k = k_n = \lceil \frac{1}{4} \log_{1/2M} n \rceil$ and using that $\sum_{i=0}^{k-1} p^{-i} \le 2(2M)^k$ for $1/2M \le p \le 1$, we deduce by elementary calculation that there exist positive constants C_1, C_2 and c_3 depending only on $c, \alpha, \beta, r, \varepsilon$, and M such that

$$\begin{aligned} P_p(n)^2 &\le C_1 n^{-1/4} + C_1 \mathbb{E}_p \exp\left[\log^\beta |K_\rho|\right] \exp\left[-c_3 \log^{\beta/(\alpha+\beta)} n\right] \\ &\le C_2 \mathbb{E}_p \exp\left[\log^\beta |K_\rho|\right] \exp\left[-c_3 \log^{\beta/(\alpha+\beta)} n\right] \end{aligned}$$

for every $1/2M \le p < p_c$ and $n \ge 1$. On the other hand, for each $u, v \in V$ and $n \ge 1$ we have by the Harris-FKG inequality that

$$\begin{aligned} \mathbf{P}_p(|E(K_u)| \ge n) &\ge \mathbf{P}_p\big(\{|E(K_v)| \ge n\} \cap \{u \leftrightarrow v\}\big) \\ &\ge \tau_p(u,v) \mathbf{P}_p(|E(K_u)| \ge n) \ge p^{d(u,v)} \mathbf{P}_p(|E(K_u)| \ge n), \end{aligned}$$

and we deduce that there exists a constant $C_3 = C_3(M,r)$ such that

$$\sup_{v\in V} \mathbf{P}_p(|K_v| \geq n) \leq \sup_{v\in V} \mathbf{P}_p(|E(K_v)| \geq n) \leq C_3 P_p(n)$$

for every $1/2M \leq p \leq 1$ and $n \geq 1$. Putting this all together, it follows that there exists a constant $C_4 = C_4(c,\alpha,\beta,r,\varepsilon,M)$ such that

$$\sup_{v\in V} \mathbf{P}_p\big(|K_v| \geq n\big) \leq C_4 \exp\left[-\frac{c_3}{2}\log^{\beta/(\alpha+\beta)} n\right] \sqrt{\mathbb{E}_p \exp\left[\log^\beta |K_\rho|\right]} \tag{4.4}$$

for every $1/2M \leq p < p_c$ and $n \geq 1$, and hence that

$$\begin{aligned}\sup_{v\in V} \mathbf{P}_p\Big(\exp\left[\log^\beta |K_v|\right] \geq x\Big) &\leq \sup_{v\in V} \mathbf{P}_p\Big(|K_v| \geq \exp\left[\log^{1/\beta} x\right]\Big)\\ &\leq C_4 \exp\left[-\frac{c_3}{2}\log^{1/(\alpha+\beta)} x\right] \sqrt{\mathbb{E}_p \exp\left[\log^\beta |K_\rho|\right]}\end{aligned}$$

for every $1/2M \leq p < p_c$ and $x \geq 1$. We integrate this bound to obtain that, since $|K_v| \geq 1$,

$$\begin{aligned}\sup_{v\in V} \mathbf{E}_p\left[\log^\beta |K_v|\right] &\leq \int_1^\infty \sup_{v\in V} \mathbf{P}_p\Big(\exp\left[\log^\beta |K_v|\right] \geq x\Big)\,\mathrm{d}x\\ &\leq C_4\sqrt{\mathbb{E}_p \exp\left[\log^\beta |K_\rho|\right]}\int_1^\infty \exp\left[-\frac{c_3}{2}\log^{1/(\alpha+\beta)} x\right]\mathrm{d}x\end{aligned}$$

for every $1/2M \leq p < p_c$. Since $\alpha+\beta<1$ this integral converges, and we obtain that there exists a positive constant $C_5 = C_5(c,\alpha,\beta,r,\varepsilon,M)$ such that

$$\mathbb{E}_p \exp\left[\log^\beta |K_\rho|\right] \leq \sup_{v\in V} \mathbf{E}_p \exp\left[\log^\beta |K_v|\right] \leq C_5\sqrt{\mathbb{E}_p \exp\left[\log^\beta |K_\rho|\right]} \tag{4.5}$$

for every $1/2M \leq p < p_c$. If $p < p_c$ then we have by sharpness of the phase transition [1,15,32] that $\mathbb{E}_p|K_\rho| < \infty$ and consequently that $\mathbb{E}_p \exp\left[\log^\beta |K_\rho|\right] \leq \mathbb{E}_p|K_\rho| < \infty$. Thus, we may safely rearrange (4.5) and deduce that there exists a constant $C_6 = C_6(c,\alpha,\beta,r,\varepsilon,M)$ such that

$$\mathbb{E}_p \exp\left[\log^\beta |K_\rho|\right] \leq \sup_{v\in V} \mathbf{E}_p \exp\left[\log^\beta |K_v|\right] \leq C_6$$

for every $1/2M \leq p < p_c$. Coupling ω_p for different values of p in the standard monotone fashion (see e.g. [22, Page 11]) and applying the monotone convergence theorem implies that this bound continues to hold at p_c, completing the proof. □

Proof of Theorems 1.2 and 1.3. Both results are immediate consequences of Theorem 4.1. □

5 Closing remarks

Remark 5.1. Suppose that G is a quasi-transitive graph satisfying ($\mathrm{HK}_{\gamma,c}$) for some $c > 0$ and $\gamma > 0$, and suppose that the simple random walk on G satisfies a bound of the form $\mathbb{P}(d(X_0,X_n) \leq$

$Cn^\nu) \geq c$ for some $1/2 \leq \nu \leq 1$, $c > 0$ and $C < \infty$. (A theorem of Lee and Peres [30] implies that such an inequality cannot hold for $\nu < 1/2$.) Then the proof of Theorem 1.2 can easily be generalized to show that critical percolation on G has no infinite clusters under the assumption that $(1-\gamma)\nu < \gamma$. We have not included the proof of this stronger result since we do not know of any examples that we can prove satisfy this condition but do not satisfy the hypotheses of Theorem 1.2. However, Tianyi Zheng has informed us that this stronger theorem *might* apply to Cayley graphs of the first Grigorchuk group, for which the optimal values of γ and ν are unknown.

Remark 5.2. More generally, a similar analysis to that discussed in Remark 5.1 shows the following: Suppose that G is a quasi-transitive graph for which there exists a symmetric stochastic matrix on G that is invariant under the diagonal action of $\mathrm{Aut}(G)$ and for which the associated random walk X satisfies ($\mathrm{HK}_{\gamma,c}$) for some $c > 0$ and $\gamma > 0$ and satisfies $\mathbb{P}(d(X_0, X_n) \leq Cn^\nu) \geq c$ for some $1/2 \leq \nu < \infty$, $c > 0$ and $C < \infty$. (One may need to take $\nu > 1$ if the walk takes long jumps.) If $(1-\gamma)\nu < \gamma$ then critical percolation on G has no infinite clusters almost surely. Long-range random walks have been a powerful tool for analyzing specific examples of groups of intermediate growth, see e.g. [17].

Remark 5.3. We expect that with a sufficiently delicate analysis one can push our method to handle all quasi-transitive graphs satisfying a return probability bound of the form $p_n(v,v) \leq \exp\left[-\omega(n^{1/2})\right]$, as well as all quasi-transitive graphs satisfying $p_n(v,v) \leq \exp\left[-\Omega(n^{1/2})\right]$ and for which the random walk has zero speed in the sense that $d(0, X_n)/n \to 0$ a.s. as $n \to \infty$. Since Conjecture 1.1 is already known in the exponential growth case and the random walk on any graph of subexponential growth has zero speed, this would allow one to extend Theorem 1.2 to the case $\gamma = 1/2$. (The fact that the random walk on a graph of subexponential growth has zero speed is an immediate consequence of the Varopoulos-Carne bound, see [31, Theorem 13.4].) On the other hand, it seems that a new idea is needed to handle the case $\gamma < 1/2$, and a solution to the following problem would be a promising next step towards Conjecture 1.1.

Problem 5.4. Extend Theorem 1.2 to quasi-transitive graphs satisfying a return probability estimate of the form $p_n(v,v) \leq \exp\left[-\Omega(n^\gamma)\right]$ for some $0 < \gamma < 1/2$.

The methods of [35] may be relevant. Note that if Grigorchuk's *gap conjecture* [20] is true, then a solution to this problem for all $0 < \gamma < 1/2$ would settle Conjecture 1.1 for all Cayley graphs of intermediate growth. Indeed, if the strong version of the conjecture is true then it would suffice to consider the case $\gamma \geq 1/5$.

Remark 5.5. We now indicate why one should not expect our proof to handle the case $\gamma < 1/2$ without substantial modification. Let G be transitive and satisfy ($\mathrm{HK}_{\gamma,c}$) for some $0 < \gamma \leq 1$ and $c_1 > 0$. As before, we let $\alpha = (1-\gamma)/\gamma$, so that $\alpha < 1$ if and only if $\gamma > 1/2$. Our proof depends upon the analytic consequences of the inequality

$$\mathbf{P}_p(|K| \geq n)^2 \leq \min_{k \geq 1} \left[\mathbf{E}_p \exp\left[-c_2 \min\left\{ k^\gamma, \frac{k}{\log^\alpha |K|} \right\} \right] + C_1 p^{-k} n^{-1/2} \right] \tag{5.1}$$

for appropriate constants $C_1 \geq 1$ and $c_2 > 0$. Let us suppose that we had the even stronger inequality

$$\mathbf{P}_p(|K| \geq n)^2 \leq \min_{x \geq 0} \left[\mathbf{E}_p \exp\left[-\frac{c_2 x}{\log^\alpha |K|} \right] + C_1 p^{-x} n^{-1/2} \right].$$

We certianly never want to take $x \geq \log_{1/p} n$ when realizing the above minimum, and since the expectation on the right hand side is decreasing in x, this inequality is weaker than the inequality

$$\mathbf{P}_p(|K| \geq n)^2 \leq \mathbf{E}_p \exp\left[-\frac{c_3 \log n}{\log^\alpha |K|} \right]$$

for appropriate choice of $c_3 > 0$. Since $\mathbf{E}_p \exp\left[-\frac{c_3 \log n}{\log^\alpha |K|}\right] \geq e^{-c_3} \mathbf{P}_p\big(|K| \geq e^{\log^{1/\alpha} n}\big)$, this inequality is, in turn, weaker than the inequality

$$\mathbf{P}_p(|K| \geq n)^2 \leq e^{-c_3} \mathbf{P}_p\Big(|K| \geq e^{\log^{1/\alpha} n}\Big). \tag{5.2}$$

Thus, any analytic consequence of the inequality (5.1) must also be a consequence of the inequality (5.2). If $\alpha \geq 1$ then $e^{\log^{1/\alpha} x} \leq x$ for every $x \geq 1$, so that any decreasing function $f : [1, \infty] \to [0, 1]$ with $f(x) \leq e^{-c_3}$ for all $x \geq 1$ trivially satisfies the inequality $f(x)^2 \leq e^{-c_3} f\big(e^{\log^{1/\alpha} x}\big)$. Thus, the inequality (5.2) does not yield any non-trivial information on the rate of decay of $\mathbf{P}_p(|K| \geq n)$ in this case. (The reader may find it illuminating to consider the constraints that (5.2) would place on the rate of decay of $\mathbf{P}_p(|K| \geq n)$ when $\alpha < 1$.) This appears to present a serious obstruction to extending our method to the case $\gamma < 1/2$.

Acknowledgments

We thank Gidi Amir and Tianyi Zheng for sharing their expertise on groups of intermediate growth. JH was supported financially by the EPSRC grant EP/L018896/1.

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